MID SEMESTER EXAMINATION - COMMUTATIVE ALGEBRA - MMATH - 29 SEPTEMBER 2010

Attempt all questions. All rings considered are commutative with 1. Total Marks - 50. Time - 3 hrs.

(1) Let A be the ring $\mathbb{Z}/12\mathbb{Z}$.
   (a) Let $\mathcal{P} = 2\mathbb{Z}/12\mathbb{Z}$ denote the prime ideal of A generated by (the coset class of) 2 in A. Show that the localization $A_{\mathcal{P}}$ is isomorphic to the ring $\mathbb{Z}/4\mathbb{Z}$. (5 marks)
   (b) Is A of finite length as a $\mathbb{Z}$-module? If yes, exhibit a composition series and compute it's length? (5 marks)

(2) (a) Compute $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ for n, m any two positive integers. (5 marks)
   (b) For M, N two A-modules and $I \subset A$ an ideal, prove that $\frac{M}{IM} \otimes_A \frac{N}{IN} \cong \frac{M \otimes_A N}{I(M \otimes_A N)}$. (5 marks)

(3) True or False? Justify your answers.
   (a) An artinian module is always noetherian. (3 marks)
   (b) A subring of a noetherian ring is always a noetherian ring. (3 marks)
   (c) If M is a noetherian A-module, then A must be a noetherian ring. (4 marks)

(4) True or false? Justify your answers.
   (a) A power of a prime ideal is always a primary ideal. (5 marks)
   (b) If A is a PID, then an ideal I of A is a primary ideal, if and only if, it is the power of a prime ideal. (5 marks)

(5) (a) Let k be a field, $A = k[x, y]$ be the polynomial ring in two variables and let $I \subset A$ be the ideal $(x).(x, y)$ (expressed as the product of two ideals). Exhibit (with proof) a minimal primary decomposition of I, write the associated primes, list which are embedded primes and which are isolated/minimal primes. (5 marks)
   (b) The same question for the ring $A = k[x, y, z]$ and the ideal $I = (x).(x, y).(x, y, z)$ (hint: use the first part to make a guess). (5 marks)